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# The $q$-Painlevé V equation and its geometrical description

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**Abstract**
We study the $q$-Painlevé V equation which can be obtained from the degeneration of the $q$-$\mathrm{P_{VI}}$ (in the form of the asymmetric $q$-$\mathrm{P_{III}}$) equation and present its geometrical description. Based on the bilinear formulation we obtain the equations for the multi-dimensional $\tau$-functions of $q$-$\mathrm{P_V}$ (in the form of nonautonomous Hirota–Miwa systems) which lives in the weight lattice of the $A_4$ affine Weyl group. This geometrical approach furnishes in a straightforward way the Miuras and the Schlesingers of $q$-$\mathrm{P_V}$.

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## The $q$-Painlevé V equation

Recent progress in the geometrical description of discrete Painlevé equations ($\mathbb{P}$s) has provided us with a most efficient tool for their classification. Starting from the 'grand scheme' [1] approach, which follows closely that of Okamoto [2] for continuous $\mathbb{P}$s, and the property of self-duality [3], which holds true for most d-$\mathbb{P}$s, we have proposed a classification approach based on affine Weyl groups [4]. These results were confirmed by the brilliant thesis of Sakai [5], who, using a slightly different approach, managed to extend them in some cases, in particular with the discovery of the elliptic-discrete $\mathbb{P}$.

Having the frame for the classification, which is provided by the degeneration pattern starting from the exceptional affine group $\mathrm{E_8}$, it is easy to spot the equations which may have been overlooked, or, at least, which have not been the object of detailed studies. One such example is the $q$-$\mathrm{P_V}$ equation:

$$\begin{aligned} y_{n-1}y_{n+1} &= \frac{(x_n - az_n)(x_n - z_n/a)}{1 - cx_n} \\ x_nx_{n+2} &= \frac{(y_{n+1} - bz_{n+1})(y_{n+1} - z_{n+1}/b)}{1 - dy_{n+1}} \end{aligned} \tag{1}$$

where $z_n = z_0q^n$ and $a$, $b$, $c$ and $d$ are constants. In fact by rescaling $x$, $y$ and $z$ one can modify $c$ and $d$, keeping their ratio constant, so there are only three parameters and one could normalize to $cd = 1$. This normalization, however, would be rather inconvenient for the continuous limit, so we keep the present one, redundant as it is. Also note that $x$ have only even indices while $y$ have only odd ones.

We must point out that $q$-$\mathrm{P_V}$ equation (1) should not be confused with the 'standard' [6] $q$-$\mathrm{P_V}$ equation:

$$(x_n x_{n+1} - 1)(x_n x_{n-1} - 1) = \frac{(x_n - a)(x_n - 1/a)(x_n - b)(x_n - 1/b)}{(1 - c x_n z_n)(1 - x_n z_n/c)}. \tag{2}$$

The latter is just a symmetric version of a much richer system [7], the continuous limit of which is $\mathrm{P_{VI}}$, as we have shown in [8]. Equation (1) was first derived in [9] as an asymmetric extension of $q$-$\mathrm{P_{II}}$ [10]. (The form asymmetric is understood here in the QRT [11] terminology.) This symmetric $q$-$\mathrm{P_{II}}$ is obtained by setting in (1) $b = a$, $d = c$ (the latter can then be scaled to any value, 1 for instance) in which case the $y$ are just odd-numbered $x$. However there exists another more direct approach based on the degeneration pattern we mentioned above. Equation $q$-$\mathrm{P_V}$ can be obtained from the degeneration through coalescence of the $q$-$\mathrm{P_{VI}}$ (asymmetric $q$-$\mathrm{P_{III}}$) equation [12] introduced by Jimbo and Sakai:

$$\begin{aligned} y_{n-1} y_{n+1} &= \frac{(x_n - a z_n)(x_n - z_n/a)}{(1 - c x_n)(1 - f x_n)} \\ x_n x_{n+2} &= \frac{(y_{n+1} - b z_{n+1})(y_{n+1} - z_{n+1}/b)}{(1 - d y_{n+1})(1 - g y_{n+1})} \end{aligned} \tag{3}$$

where $z_n = z_0 q^n$ and $a$, $b$, $c$, $d$, $f$ and $g$ are nonzero constants satisfying the constraint $cf = dg$. Taking the limit $f \to 0$, $g \to 0$ we obtain indeed equation (1). From the relation of $q$-$\mathrm{P_V}$ to equation (3) it is clear that one can obtain its Lax pair by implementing the appropriate limit on the Lax pair of (3) presented by Jimbo and Sakai. The reason equation (1) is called $q$-$\mathrm{P_V}$ is based on the continuous limit. Taking $q = 1 + \epsilon$, $y = x + z + \mathcal{O}(\epsilon)$, $a = 1 + \epsilon a_1$, $b = -1 + \epsilon b_1$, $c = \epsilon c_1 + \epsilon^2 c_2$ and $d = -\epsilon c_1 + \epsilon^2 c_2$ we find indeed at $\epsilon \to 0$ for the quantity $w = y/x$

$$w'' = w'^2 \left( \frac{1}{2w} + \frac{1}{w-1} \right) - \frac{w'}{z} + \frac{(w-1)^2}{z^2} \left( \frac{a_1^2}{2} w - \frac{b_1^2}{2w} \right) - \frac{c_2 w}{z} - \frac{c_1^2 w(w+1)}{8(w-1)} \tag{4}$$

i.e. $\mathrm{P_V}$ in canonical form.

We now turn to the geometrical description of $q$-$\mathrm{P_V}$.

### *The $A_4$ weight lattice and its geometry*

From the geometrical description of discrete $\mathbb{P}$s we have presented in [3, 7], it has become clear that the pertinent space is the weight lattice of some affine Weyl group, i.e. the dual of the root system. It turns out that for the $q$-$\mathrm{P_V}$ equation we are analysing in this paper the space we must consider is the one associated with $A_4$. Although this space is invariant under the action of the symmetries of $A_4$ there exists no way to define an invariant orthonormal basis for the weight lattice. We could of course have used a noninvariant (non-normal) orthogonal basis, just as we did in [7], but instead here we choose to use a description within a five-dimensional space, or rather a hyperplane of such a space. Taking the hyperplane of $\mathbb{Z}^5$ where the sum of all coordinates vanishes, we can describe the points of the weight lattice of $A_4$, where the $\tau$-functions live. These points are generated by the five vectors (only four of which are independent) having one coordinate equal to 4 and the four others equal to $-1$. One can normalize our space by deciding that the norm of a unit vector in any of the five directions of $\mathbb{Z}^5$ is $1/\sqrt{20}$, so our basic vectors have length 1. Note that the sum of the five basic vectors is zero, and the dot product of any two distinct ones is $-1/4$. This choice allows us to consistently orient them. The weight lattice, thus, consists of all points in $\mathbb{Z}^5$ that satisfy the following conditions:

- the sum of all coordinates is zero and
- all five coordinates are congruent to each other with respect to 5.

The origin does satisfy these requirements. In the case of $A_4$ its nearest neighbours (NNs) are just the endpoints of the five basic vectors (and their opposites). Though the adjective 'nearest' is not appropriate for these vectors, which are actually the smallest ones, we will call them NVs for 'nearest-neighbour-connecting vectors', a shorthand the reason for which will soon become obvious. Having established that each $\tau$ has 10 NNs, we turn to next-nearest-neighbours (NNNs). These can be reached by moving away from this $\tau$ by a vector which is as small as possible a sum of NVs. This turns out to be the case if we add two NVs since their dot product is negative, namely $-1/4$. So the squared length of such a next-nearest-neighbour-connecting vector NNV is $3/2$. There are just ten such consistently oriented vectors, of the form $(3, 3, -2, -2, -2)$, where of course the two 3s can be at *any* two positions. There are 20 NNNs of the origin, at the endpoints of these ten vectors and their opposites. The difference of two NVs has squared length $5/2$, a typical one being $(-5, 5, 0, 0, 0)$. There are also ten such vectors, up to a sign, but note that these *cannot* be consistently oriented. They connect $\tau$ in next-next-nearest-neighbour (NNNN) positions. Any of these NNNVs is the difference of exactly one pair of NVs, (e.g. $(-1, 4, -1, -1, -1)$ and $(4, -1, -1, -1, -1)$ for the NNNV $(-5, 5, 0, 0, 0)$ mentioned above), and it is orthogonal to the other three NVs. So by adding to it (in either orientation) any of the latter we obtain vectors of squared length $7/2$. However, we obtain only 30 rather than 60 such vectors since each is obtained in two such ways. In fact these squared length $7/2$ vectors are the sum of two distinct NVs, minus a third one distinct from the first two. These vectors can be consistently oriented and a typical one is $(-6, 4, 4, -1, -1)$.

### *The bilinear form of the $q$-$P_{\mathrm{V}}$*

We start by introducing the nonlinear variables (for which we will use the symbols $X$ or $Y$) and assume that they are defined at points of the lattice which are midpoints between one $\tau$ and one of its NNNs, or equivalently one $\tau$ and one of its NNNNs (and, also, the midpoint of vectors of squared length $7/2$). Indeed, if we take one $\tau$ and any two of its NNs (which are either in NNN or NNNN respective position depending on whether we move along NVs of the opposite or same orientation) and complete the parallellogram (in fact a rhombus), the centre of this rhombus is the site of our nonlinear variable. The short (squared length $3/2$) and long (squared length $5/2$) diagonals define two pairs of $\tau$ which are uniquely defined for each such site. A typical site can be $(-3, 2, 2, -\frac{1}{2}, -\frac{1}{2})$, for instance. The pair of $\tau$ in NNN relative position around it is $\{(-4, 1, 1, 1, 1), (-2, 3, 3, -2, -2)\}$ while the pair in NNNN position is $\{(-3, 2, 2, 2, -3), (-3, 2, 2, -3, 2)\}$. Note that, on purpose, we chose this point so that its squared distance to the origin is neither $3/8$ nor $5/8$ but $7/8$, so it is also the midpoint of a vector of length $7/2$ originating at the origin, namely the one mentioned at the end of the paragraph above. Moreover there are exactly two more pairs of $\tau$ of squared distance $7/2$ such that our site is their midpoint, namely $\{(-1, 4, -1, -1, -1), (-5, 0, 5, 0, 0)\}$ and the one obtained from it by permuting the second and the third coordinates of both sites.

The next step is to relate the nonlinear variable $X$ to the $\tau$. Each $X$, at the centre of a rhombus, is the ratio of the product of the $\tau$ at the end of the long diagonal to the product of the $\tau$ at the end of the short diagonal. For instance

$$X\left(-3, 2, 2, -\frac{1}{2}, -\frac{1}{2}\right) = \frac{\tau(-3, 2, 2, 2, -3)\tau(-3, 2, 2, -3, 2)}{\tau(-4, 1, 1, 1, 1)\tau(-2, 3, 3, -2, -2)}. \tag{5}$$

We can also use the three pairs of $\tau$ separated by a distance of square $7/2$ and having $X$ as midpoint to find more expressions of $X$. For any of these three pairs one can express $X$ in terms of the product of these $\tau$ and the product of the $\tau$ in NNN relative position. From previous experience we expect this relation to depend explicitly on the position vector $\overrightarrow{O'X}$ (note here that the origin $O'$ of this vector need not coincide with the origin of coordinates: it may well be shifted by five arbitrary numbers of zero sum), through its dot product with a characteristic vector of the equation. In this case, however, contrary to all previously studied ones, there are *two* oriented vectors in the problem, one that depends only on $X$, namely the one relating the two $\tau$ in NNN position (the vector relating the two $\tau$ in NNNN position cannot be consistently oriented and thus its dot product with $\overrightarrow{O'X}$ may not appear) and a further one that depends on the particular pair considered, i.e. the relevant squared length $7/2$ vector. In order to make things clearer we take a specific example. Around $X(-3, 2, 2, -\frac{1}{2}, -\frac{1}{2})$ let us denote by $A$ the quantity $\overrightarrow{O'X} \cdot \overrightarrow{(-2, -2, -2, 3, 3)}$ where the second vector is the one joining the two sites in NNN position. We have for instance the pair of points $(0, 0, 0, 0, 0)$ and $(-6, 4, 4, -1, -1)$ at squared distance $7/2$. Let us denote by $B$ the relevant quantity $\overrightarrow{O'X} \cdot \overrightarrow{(-6, 4, 4, -1, -1)}$, the latter vector being positively oriented. We thus expect

$$X\left(-3, 2, 2, -\frac{1}{2}, -\frac{1}{2}\right) = Q^{\alpha A+\beta B}\frac{\tau(0, 0, 0, 0, 0)\tau(-6, 4, 4, -1, -1)}{\tau(-4, 1, 1, 1, 1)\tau(-2, 3, 3, -2, -2)} - Q^{\gamma A+\delta B} \tag{6}$$

for some $Q$ and appropriate values of $\alpha, \ldots, \delta$ (of course an overall scaling of the latter coefficients can be absorbed in the definition of $Q$). Similar equations can be obtained for the two other choices of pair of two $\tau$ at squared distance $7/2$ around $X(-3, 2, 2, -\frac{1}{2}, -\frac{1}{2})$ with the same $A$ but different $B$s for each pair.

Comparing (5) to (6) allows us to obtain equations relating the product of the two NNN $\tau$, the product of the two NNNN $\tau$ and the product of two $\tau$ at squared distance $7/2$ around the same point. These equations are nonautonomous Hirota–Miwa [13] equations, the general form of which is

$$\tau_{\mathrm{NNNN}}\tau_{\mathrm{NNNN}} = Q^{\overrightarrow{O'X}\cdot(\alpha\overrightarrow{\mathrm{NNV}}+\beta\overrightarrow{\tau_-\tau_+})}\tau_-\tau_+ - Q^{\overrightarrow{O'X}\cdot(\gamma\overrightarrow{\mathrm{NNV}}+\delta\overrightarrow{\tau_-\tau_+})}\tau_{\mathrm{NNN}}\tau_{\mathrm{NNN}} \tag{7}$$

where $\overrightarrow{\mathrm{NNV}}$ is the positively oriented NNV relating the two $\tau$ in NNN position around $X$ and $(\tau_-, \tau_+)$ form any of the three pairs at squared distance $7/2$ around $X$ such that the vector joining them in this order is positively oriented.

This system is highly overdetermined, and thus one may wonder whether it is consistent. Clearly, it will be consistent only for the appropriate values of $\alpha, \ldots, \delta$. By implementing the consistency requirement we find two independent constraints, which can be, for instance, written as

$$\alpha - \beta - 2\gamma = 0 \qquad 2\beta - \gamma - \delta = 0. \tag{8}$$

The existence of only two constraints, rather than three, is surprising. In fact we expected the consistency requirement to determine completely the ratios of the quantities $\alpha, \ldots, \delta$ (as we remarked above, an overall factor can be absorbed in $Q$). It turns out that the reason why only two constraints appear is the existence of a gauge transformation of the $\tau$ that leaves the overall picture invariant but modifies the values of the $\alpha, \ldots, \delta$. This gauge is $\tau_{n_i} \to Q^{\kappa\Sigma n_i^3}\tau_{n_i}$ where the $n_i$ are the five coordinates of the site of this $\tau$ in (the relevant hyperplane of) $\mathbb{Z}^5$. The existence of such a 'cubic' gauge seems to be rather special to $A_4$. In most other Weyl-group-based grand schemes we studied before [3,7], such a gauge could not be invariant under all the symmetries of the group. Of course this gauge also changes the definition of $X$ but this will be absorbed in the relation (that we have not yet given) between the $X$ variable of (5) and the $x$ variable appearing in (1). Using the gauge, we can for instance choose $\beta = 0$, and, normalizing $\delta$ to 1, we have $\gamma = -1$, $\alpha = -2$. We can rewrite (7) as

$$\tau_-\tau_+ = Q^{2\,\overrightarrow{O'X}\cdot\overrightarrow{\mathrm{NNV}}}\tau_{\mathrm{NNNN}}\tau_{\mathrm{NNNN}} + Q^{\overrightarrow{O'X}\cdot(\overrightarrow{\mathrm{NNV}}+\overrightarrow{\tau_-\tau_+})}\tau_{\mathrm{NNN}}\tau_{\mathrm{NNN}}. \tag{9}$$

The system of equations (9) describes completely the evolution of the multivariable $\tau$-function in $A_4$. They are, in fact, the bilinear forms of the $q$-$\mathrm{P_V}$ equation.

*Contiguity relations and the nonlinear equations*

We proceed now to derive the nonlinear evolution equations for the variable $X$ introduced above. In what follows, whenever there is no ambiguity, we will use the name of a nonlinear variable to mean the point where this variable is defined.

The NNNV, long diagonal of the rhombus of centre $X$, is orthogonal to three NVs. So the vector joining one of its endpoints to the point obtained by translating the other endpoint by any of these three NVs has squared length 7/2. Its midpoint is thus the location of some nonlinear variable, say $Y$. By construction, the vector $\overrightarrow{XY}$ is half the corresponding NV. There are thus three directions along which $X$ has neighbours at distance 1/2. Of course $X$ has such a neighbour in both orientations for each direction, so there are six altogether. Still, the orientation is consistently defined so we can distinguish positively shifted and negatively shifted $Y$s. So $Y_1^+ = (-1, \frac{3}{2}, \frac{3}{2}, -1, -1)$, $Y_1^- = (-5, \frac{5}{2}, \frac{5}{2}, 0, 0)$, and similarly for the indices 2 and 3 which refer to the order of the coordinate of value 4 in the NV. (It turns out that, in that case, it is indeed the three first which are relevant since the NNNV through $X$ is $(0, 0, 0, 5, -5)$.) By construction, both the endpoints of the NNNV around $X$ belong to pairs of $\tau$ at square distance 7/2 around each of the $Y$ (indeed we just defined $Y$ as the midpoint of one diagonal of a rectangle constructed on this NNNV, but it is of course also the midpoint of the other diagonal). This fact is not immediately useful to us, but since $X$ and $Y$ play exactly the same role, it follows that one can predict that the endpoints of the NNNV through each $Y$ belong to the pairs of $\tau$ at square distance 7/2 around $X$. For instance, for $Y_1^+$ the NNNV relates $(-1, 4, -1, -1, -1)$ to $(-1, -1, 4, -1, -1)$, and for $Y_1^-$, $(-5, 5, 0, 0, 0)$ and $(-5, 0, 5, 0, 0)$, all four of these points being at squared distance 7/8 from $X$. By inspection, one can check that the (oriented) NNV around, say, $Y_1^-$, relates $(-4, 1, 1, 1, 1)$ to $(-6, 4, 4, -1, -1)$. So

$$Y_1^- = \frac{\tau(-5,5,0,0,0)\tau(-5,0,5,0,0)}{\tau(-4,1,1,1,1)\tau(-6,4,4,-1,-1)}. \tag{10}$$

Similarly, we have for $Y_2^+$ at $(-\frac{7}{2}, 4, \frac{3}{2}, -1, -1)$

$$Y_2^+ = \frac{\tau(-1,4,-1,-1,-1)\tau(-6,4,4,-1,-1)}{\tau(-5,5,0,0,0)\tau(-2,3,3,-2,-2)} \tag{11}$$

the two $\tau$ in the numerator being among the six at squared distance 7/8 of $X$. Computing the product $Y_1^- Y_2^+$ we find, after some simplifications,

$$Y_1^- Y_2^+ = \frac{\tau(-5,0,5,0,0)\tau(-1,4,-1,-1,-1)}{\tau(-4,1,1,1,1)\tau(-2,3,3,-2,-2)}. \tag{12}$$

We can recognize that the denominator is just that of $X$ and both $\tau$ at the numerator form a pair at squared distance 7/2 of midpoint $X$. We can now use equation (9) around $X$ to express the numerator of the above expression, and find

$$Y_1^- Y_2^+ = Q^{2A}X + Q^{A+\overrightarrow{O'X}\cdot\overrightarrow{(4,4,-6,-1,-1)}} \tag{13}$$

with the same $A$ as in (6). This is a contiguity relation in the isosceles triangle $Y_1^- X Y_2^+$ in which the summit $X$ plays a special role. Note that the angle at summit $X$ is acute, having a cosine equal to $1/4$, since it is the angle of one positively oriented NV to a negatively oriented one. We have similar Miura relations in the triangles $Y_3^- X Y_2^+$ and $Y_3^- X Y_1^+$:

$$Y_3^- Y_2^+ = Q^{2A} X + Q^{A + \overrightarrow{O'X} \cdot \overrightarrow{(-6,4,4,-1,-1)}} \tag{14}$$

where the last dot product is just what we called $B$ just above, and similarly

$$Y_3^- Y_1^+ = Q^{2A} X + Q^{A + \overrightarrow{O'X} \cdot \overrightarrow{(4,-6,4,-1,-1)}}. \tag{15}$$

By multiplying together, side by side, equations (13) and (15), and dividing by (14) we find the equation

$$Y_1^- Y_1^+ = \frac{Q^{2A}(X + Q^{C-A})(X + Q^{D-A})}{X + Q^{B-A}} \tag{16}$$

with obvious definitions for $C, D$. Note that the NV along the direction $\overrightarrow{XY_1}$, namely $\overrightarrow{U} \equiv \overrightarrow{(4, -1, -1, -1, -1)}$, has the same dot product 1 with the two vectors of squared length $7/2$ entering $C$ and $D$ and moreover it is just their half-sum. Its dot product with the one entering $B$ is $-3/2$, and its dot product with the NNV entering $A$ is $-1/2$.

One can obtain a similar relation around $Y_1^+$, relating this point to $X$ and the point $X^{++}$ obtained by translating $X$ by the *full* NV $\overrightarrow{U}$. Note that we denote this point with two upper indices + since it is the translation of $Y_1^+$ by the elementary propagation step, namely one-half $\overrightarrow{U}$. We find around $Y_1^+$ the NNV $\overrightarrow{(-2, 3, 3, -2, -2)}$ and three new vectors of squared length $7/2$ but the dot products with $\overrightarrow{U}$ are the same in the objects appearing in numerator and denominator respectively. Formally

$$X X^{++} = \frac{Q^{2E}(Y_1^+ + Q^{G-E})(Y_1^+ + Q^{H-E})}{Y_1^+ + Q^{F-E}}. \tag{17}$$

Since the lattice is invariant by translation by any *full* NV, the equations around the translation of $X$ or $Y_1^+$ by any multiple of $\overrightarrow{U}$ have exactly the form of (16) or (17) respectively, with the same squared $7/2$ vectors. The only difference is in the position vector, which is incremented by integer multiples of the unit length vector $\overrightarrow{U}$. So in the equation around $X^{++}$, say, the increments of $A$, $B$, $C$ and $D$ are $-1/2$, $-3/2$, 1 and 1 respectively compared to their values around $X$. The same will be true of $E$, $F$, $G$ and $H$ respectively if we go to the point $Y_1^{+++}$ instead of $Y_1^+$.

We now introduce the variable transformation

$$X = x Q^{(3B-A)/4} \qquad Y_1 = y Q^{(3F-E)/4} \tag{18}$$

where the $A$, $B$s etc must be considered as the dot product of the position vector from $O'$ to the point considered, with the appropriate NNVs or squared length $7/2$ vectors. Then equation (16) becomes

$$Y_1^- Y_1^+ = Q^{(5A+B)/2} \frac{(x + Q^{C-3(A+B)/4})(x + Q^{D-3(A+B)/4})}{Q^{(3A-B)/4} x + 1}. \tag{19}$$

Since $Y_1^-$ and $Y_1^+$ are symmetrical with respect to $X$, the renormalization factor of the lhs is the dot product of $2\overrightarrow{O'X}$ with the vector that determines $(3F - E)/4$, but, since $3F - E$ is determined by the same vector $\overrightarrow{(-16, -6, -6, 14, 14)}$ as $5A + B$, this factor cancels out with the factor on the rhs, and we obtain

$$y^- y^+ = \frac{(x + Q^{C-3(A+B)/4})(x + Q^{D-3(A+B)/4})}{Q^{(3A-B)/4} x + 1}. \tag{20}$$

Similarly, (17) becomes

$$xx^{++} = \frac{(y^+ + Q^{G-3(E+F)/4})(y^+ + Q^{H-3(E+F)/4})}{Q^{(3E-F)/4}y^+ + 1}. \tag{21}$$

Equations (20) and (21) are valid not only around the specific points, $x$ and $y^+$ respectively, but at all points of the lattice obtained from the latter through translations by an integer multiple of the NV $\overrightarrow{U}$. This is ensured by the definitions of $A, B, \ldots, G$ which were appropriately made above. We can remark here that the quantities $3A - B$ and $3E - F$ that appear in the denominators of (20) and (21) are invariant under translation by multiples of $\overrightarrow{U}$, while the quantities appearing in the numerators increase by $5/2$ upon translation by one $\overrightarrow{U}$. Moreover the sum of the quantities appearing in the exponents in each numerator is the dot product of the relevant position vector with exactly $5\overrightarrow{U}$. We can thus introduce the independent variable $n$, related to the number of steps of *one half* $\overrightarrow{U}$, so the $x$s and the $y$s have indices of different parities. Denoting by $z_n$ the quantity $Q^{\frac{5}{2}\overrightarrow{O'X}\cdot\overrightarrow{U}}$ (or $Q^{\frac{5}{2}\overrightarrow{O'Y_1}\cdot\overrightarrow{U}}$) we have $z_n = z_0q^n$ provided we choose $q = Q^{5/4}$. We thus recover exactly equations (1)

$$y_{n-1}y_{n+1} = \frac{(x_n - az_n)(x_n - z_n/a)}{1 - cx_n}$$

$$x_nx_{n+2} = \frac{(y_{n+1} - bz_{n+1})(y_{n+1} - z_{n+1}/b)}{1 - dy_{n+1}}$$

provided we denote by $a$, $b$, $c$ and $d$ the constant quantities $-Q^{(C-D)/2}$, $-Q^{(G-H)/2}$, $-Q^{(3A-B)/4}$ and $-Q^{(3E-F)/4}$ respectively. With the choices we have made it turns out that $cd = 1$, but this is not invariant under a coordinate transformation.

*The oblique equation*

Equation (1) is not the only one that can be found on this space. Other pathways can be considered, leading to more equations. For instance, after moving from $Y_1^-$ to $X$ by one-half $\overrightarrow{U}$, instead of going in the same direction to $Y_1^+$ one could go from $X$ to $Y_2^+$ with one-half of the NV with component equal to 4 in second position. The relation between these three points has already been given as equation (13). The point $Y_2^+$ has half-integer coordinates at positions 1 and 3, so the direction of $\overrightarrow{U}$ as well as the one with 4 in the third position are not allowed, but we can use an NV with 4 in the fourth or fifth position (or second, of course, but we do not want to keep going in the direction from which we arrived). Let us choose the fifth. We thus reach the point $V^{++} = (-4, \frac{7}{2}, 1, -\frac{3}{2}, 1)$. Now this point is *not* the translation of the point $Y_1^-$ we started from by a full number of NVs. Indeed the integer/half-integer characters of the coordinates of third and fourth positions are interchanged but that of the coordinates of first, second and fifth positions is the same as for $Y_1^-$. So from $V^{++}$ the direction $\overrightarrow{U}$ is again allowed, and we reach $(-2, 3, \frac{1}{2}, -2, \frac{1}{2})$, from which the direction with the 4 in second position is allowed, leading to $(-\frac{5}{2}, 5, 0, -\frac{5}{2}, 0)$. Finally from the latter point the direction with 4 in the fifth position is again allowed, to $(-3, \frac{9}{2}, -\frac{1}{2}, -3, 2)$. This point *is* the translation of the point $Y_1^-$ we started from, by a full number of NVs, namely the vector $(2, 2, -3, -3, 2)$. This is a negatively oriented NNV, as expected since the sum of three distinct NVs is just the opposite of the sum of the two others. From then on the same motion can be repeated indefinitely, since the lattice is invariant under this translation. This trajectory in the lattice has an approximate threefold periodicity, present in the motion along the three directions 1, 2, 5, 1, 2, 5, etc, but the *exact* periodicity is of order 6. Here to avoid proliferation of names of variables we call them all $x$ and with the appropriate change of variables (different from (18):

for instance, $X = xQ^{\overrightarrow{O'X}\cdot\overrightarrow{(4,4,-6,-1,-1)}-A})$ we obtain

$$x_{n-1}x_{n+1} = z_n q^{\phi_n}(x_n + 1). \tag{22}$$

The overall propagation direction, of course, is given by the negatively oriented NNV $(2, 2, -3, -3, 2)$, but there are six steps of $n$ in one full period. We find that $z_n = z_0 q^n$ where here one should take $q = Q^{-5/12}$. The phase $\phi_n$ has the form $\phi_n = p(-1)^n + rj^n + tj^{2n}$ with $j$ a cube root of unity and $p, r, t$ three arbitrary constants depending on the position of $O'$. One sees that the exact periodicity of $\phi$ is of order 6. There is however an underlying partial symmetry of order 3, reflecting that of the propagation direction at each step. The underlying period 2 symmetry is related to the integer/half-integer characters of the coordinates of positions 3 and 4. This equation was first presented in [9] where the connection with equation (1) was already announced (but without explicit proof).

*Summary*

In this paper we have analysed the $q$-$\mathrm{P_V}$ equation

$$y_{n-1}y_{n+1} = \frac{(x_n - az_n)(x_n - z_n/a)}{1 - cx_n}$$
$$x_n x_{n+2} = \frac{(y_{n+1} - bz_{n+1})(y_{n+1} - z_{n+1}/b)}{1 - dy_{n+1}}$$

and its geometrical structure. The $\tau$-function for this discrete Painlevé equation is a multi-dimensional object depending on the independent variable and the parameters of the equation

$$\tau(n; a, b, c, d)$$

with one constraint between the parameters, for instance $cd = 1$. The $\tau$-function lives in a four-dimensional space which is that of the weights of the affine group $A_4$. The bilinear formalism led (as is expected in a 'grand scheme' formulation) to a bilinear form of $q$-$\mathrm{P_V}$ as a system of nonautonomous Hirota–Miwa equations of the general form

$$\tau_+\tau_- = \lambda^\alpha \overline{\tau}\underline{\tau} + \lambda^\beta \tilde{\tau}\underset{\sim}{\tau}$$

where the 'bar' and 'tilde' indicate two orthogonal directions in the lattice (up and down positions meaning opposite directions of evolution). The parameters $\alpha$ and $\beta$ depend linearly on $n$ and the parameters of the equation. Introducing the nonlinear variables we obtained the Miura relations which relate three variables $X$, $Y$, $W$ which occupy the vertices of an isosceles triangle, with $X$ at the summit. The general form of the Miura is

$$YW = \lambda^\gamma X + \lambda^\delta$$

where $\gamma$ and $\delta$ also depend linearly on $n$ and the parameters of the equation. In order to obtain the evolution equations one must combine three of these triangles (which do not lie on the same plane) and eliminate the auxiliary variable, say $W$. The final form of $q$-$\mathrm{P_V}$ is obtained with the appropriate gauge of $x$, $y$ and the introduction of the independent variable $z$ depending exponentially on $n$. The geometrical approach introduced in this paper made also possible the derivation of an equation that lives in the same space but follows a different evolution path.

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